Displacing critical points from the boundary

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Let M be an $n+1$ ($n \geq 1$) manifold with boundary ∂M, and let $f\colon M \to \mathbb{R}$. For $p \in \mathrm{Int}M$, smoothness, derivatives are defined as usual. For $p \in \partial M$, let $(U, x^1, \ldots, x^{n+1})$ be a chart around p with boundary mapping to $x^1 = 0$ and interior to $x^1 > 0$ and p mapping to 0. The pushforward of f is smooth at the origin if it extends to a smooth function $\tilde{f}$ on a neighbourhood V of the origin. The derivative at the origin is defined as the derivative of $\tilde{f}$. By continuity, the derivative at 0 is the limit of the derivatives from the upper half space, therefore this derivative is independent of the extension $\tilde{f}$. Similarly one can talk of the Hessian at critical points (points where the derivative vanishes).

Theorem 1. *Suppose $f\colon M \to \mathbb{R}$ is smooth and $\mathrm{grad} f$ is tangential along ∂M. Suppose $p \in \partial M$ is a critical point of f, i.e., the $df_p = 0$, then there is a chart $(U, x^1, \ldots, x^{n+1})$ around p such that p maps to $0, \partial M$ lies in $x^1 = 0, \mathrm{Int}M$ lies in $x^1 > 0$ and*

$$f = f(p) - (x^1)^2 - \cdots - (x^k)^1 + (x^{k+1})^2 + \cdots + (x^{n+1})^2$$

on U where k is the index of the Hessian of f at p.

Proof. This proof is from [1]. First, by choosing an arbitrary chart around p, we may take our domain to be a neighbourhood U of 0 in $\mathbb{R}^{n+1}_{x^1 \geq 0}$. Extend f to a convex neighbourhood V, to be precise we may suppose that V is a ball around the origin, of 0 and call this extension f. By Hadamard lemma, we can find smooth functions g_i such that on U,

$$f = f(0) + x^1 g_1 + \cdots + x^{n+1} g_{n+1}.$$

Once again, by Hadamard lemma, we may write for $j \geq 2$,

$$g_j(x^1, \ldots, x^{n+1}) = g_j(0, x^2, \ldots, x^{n+1}) + x^1 h_j(x^1, \ldots, x^{n+1})$$

so we may assume that the $g_j, j \geq 2$ do not depend on x^1.

The gradient being tangential to the boundary means that

$$\frac{\partial f}{\partial x^1}(0, x^2, \ldots, x^{n+1}) = 0$$

so, by applying Hadamard lemma again, we may write

$$g_1(x^1, \ldots, x^{n+1}) = x^1 h_1(x^1, \ldots, x^{n+1}).$$

Putting all these together, we get

$$f(x) = f(0) + (x^1)^2 h_1 + \sum_{j \geq 2} x^j g_j(0, x^2, \ldots, x^{n+1}).$$

0 is a nondegenerate critical point, so $h_1(0) \neq 0$, so we may shrink V and assume $h_1 \neq 0$ on V. Now, replace x^1 with $x^1/\sqrt{\pm h_1}$, so that $h_1 = \pm 1$. The remaining terms do not depend on x^1 and we may use the standard techniques from [2] to reduce it to a diagonal quadratic form.

The important thing to observe is that the boundary is still mapped to the $x^1 = 0$ section and interior to the $x^1 > 0$ region, because the x^1 coordinate is changed only at the first step and all further transformations modify $x^j, j \geq 2$. □

Now, suppose f is defined on some neighbourhood of 0 as above, then we know that 0 is a critical point. Let us say that U contains an $B_{2\epsilon}(0)$ and let $a = (a^1, \ldots, a^{n+1})$ be a point within ϵ of 0. Consider the bump function

$$\phi(x) = \begin{cases} \epsilon^2 \exp(\frac{-1}{1-(r_a/\epsilon)^2}), & 0 \leq r_a = |x-a| \leq \epsilon. \\ 0, & r_a > \epsilon. \end{cases} \tag{1}$$

It is well known that ϕ is a smooth function. Consider the function $f + \phi$ defined on U where $f = f(0) - (x^1)^2 - \cdots - (x^k)^1 + (x^{k+1})^2 + \cdots + (x^{n+1})^2$. We look at the critical points of f. First we look at the derivative of ϕ. On $r_a < \epsilon$ we have

$$\begin{aligned} \frac{\partial \phi}{\partial x^i} &= \epsilon^2 \exp(\frac{-1}{1-(r_a/\epsilon)^2})(1-(r_a/\epsilon)^2)^{-2}(-2\frac{x^i - a^i}{\epsilon^2}) \\ &= -2\beta(x^i - a^i) \end{aligned}$$

where $\beta = \exp(\frac{-1}{1-(r_a/\epsilon)^2})(1-(r_a/\epsilon)^2)^{-2} \geq 0$. On the region $r_a \geq \epsilon, \phi = 0$, so $f + \phi = f$ and the derivates, critical points do not change. On $r_a < \epsilon$, the critical points satisfy

$$\pm 2x^i = 2\beta(x^i - a^i)$$

so, we must have

$$x^i = \begin{cases} \frac{\beta}{\beta+1} a^i, & 1 \leq i \leq k \\ \frac{\beta}{\beta-1} a^i, & k+1 \leq i \leq n+1. \end{cases} \tag{2}$$

Note that β is a function of x and a. Look at the function

$$h = \frac{\exp(\frac{-1}{1-x^2})}{(1-x^2)^2}$$

defined on $(-1,1)$ to find bounds on β.

It is easy to see that $h \to 0$ near ± 1 and therefore bounded. A quick calculation shows that h' is zero at $0, \pm 1/\sqrt{2}$ and computing the values shows that $0 \leq h \leq .75$ on $[-1,1]$. The same bounds apply to β Note that since $\phi = 0$ on $r_a \geq \epsilon$, the critical points in this region are those of f, but f has only one critical point which is 0 (and $r_a(0) < \epsilon$ by assumption on a).

Therefore, if $f + \phi$ has a critical point in U, then it satisfies the equation above, in particular, $\beta \neq 0$.

If $k \geq 1$, then choose a so that $a^1 < 0$. It then follows that whatever critical point in U must satisfy the equation above, thus $x^1 = \frac{\beta}{\beta+1} a^1 < 0$ as $0 < \beta/(\beta+1)$.

On the other hand, if $k = 0$, then we simply choose a so that $a^1 > 0$. Then any critical point in U has $x^1 = \frac{\beta}{\beta-1} a^1 < 0$ as $\beta/(\beta-1) < 0$.

Therefore, depending on the index, we can choose a so that the critical point (if it exists) lies in the region $x^1 < 0$. Therefore, $f + \phi$ has one lesser critical point (because we displaced the critical point to outside the manifold). Note however that this procedure applies only to critical points on the boundary having the local description as in the Morse lemma above.

References

[1] Morse theory on Manifolds without boundary, Maciej Borodzik et al., Algebraic & Geometric Topology, Volume 16 (2016), pdf: https://core.ac.uk/download/pdf/78475974.pdf

[2] Milnor's Morse Theory